Calculations show that condition (4.5) holds for PM's which are stable to a first approximation (the calculations are particularly simple if, as in /10/, the motion of the body is described by using the canonically conjugate Anduaille variables). Consequently, these PM's are in fact orbitally stable.

In accordance with Sect.4, for the first of these PM's there are just two families of asymptotic motions, while there are no motions asymptotic to the second PM.

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THE PARTIAL STABILITY OF MOTION*

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It is proved that the problem of stability (asymptotic stability) with respect to some of the variables, for a linear system with periodic analytic coefficients, is equivalent to the same problem with respect to all the variables, either for the same system or for an auxiliary linear system with periodic but not necessarily continuous coefficients, in less dimensions than the original system. A constructive procedure is described for constructing this auxiliary system, and the necessary and sufficient conditions are established for partial stability (asymptotic stability), generalizing the results of the Floquet-Lyapunov theory.

It is shown that the class of non-linear systems for which the problem of partial stability is solvable by linear approximation may be enlarged if, instead of the linear part of the original (non-linear) system, one considers a specially constructed linear approximating system which is equivalent to a certain non-linear subsystem of the original system. Constructive procedures are described for constructing such auxiliary systems, and a theorem on partial stability is proved. Well-known theorems on stability in the Lyapunov-critical cases are extended.

1. Formulation of the problem of the stability of a linear system with periodic coefficients. We consider a linear system of ordinary differential equations of perturbed motion:

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$$\frac{dx_i}{dt} = \sum_{j=1}^{n} A_{ij} x_j \quad (i = 1, \ldots, n)$$

$$\mathbf{x} = (x_1, \ldots, x_n) = (y_1, \ldots, y_m, z_1, \ldots, z_p) = (\mathbf{y}, \mathbf{z})$$

$$m > 0, \quad p \quad 0, \quad n = m + p$$

or, in y, z variables

$$\frac{dy_i}{dt} = \sum_{k=1}^{m} a_{ik} y_k + \sum_{l=1}^{p} b_{il} z_l \quad (i = 1, \ldots, m) \tag{1.1}$$

$$\frac{dz_j}{dt} = \sum_{k=1}^{m} c_{jk} y_k + \sum_{l=1}^{p} d_{jl} z_l \quad (j = 1, \ldots, p)$$

The coefficients A_{ij}, a_{ik}, b_{il}, c_{jk}, d_{jl} are T-periodic analytic functions of $t \in [0, \infty)$.

We shall consider the problem of stability (asymptotic stability) with respect to $y_1, \ldots, y_m$ /1, 2/ of the unperturbed motion $\mathbf{y} = 0$, $\mathbf{z} = 0$ of system (1.1).

2. Auxiliary proposition. Let us express system (1.1) in vector form:

$$\mathbf{y}^{\cdot} = A\mathbf{y} + B\mathbf{z}, \quad \mathbf{z}^{\cdot} = C\mathbf{y} + D\mathbf{z} \quad (\mathbf{x}^{\cdot} = A^*\mathbf{x}) \tag{2.1}$$

where A^*, A, B, C, D are matrix-valued functions of $t \in [0, \infty)$ of appropriate dimensions, and consider the matrices

$$G_1 = L_1, \ G_2 = \{L_1, L_2\}, \ldots, G_j = \{L_1, \ldots, L_j\} \quad (j = 3, \ldots, p + 1)$$

whose elements are determined by the relationships

$$L_1 = B^T, \ldots, L_j = L_{j-1}^{\cdot} + D^T L_{j-1} \ (j = 2, \ldots, p + 1)$$

($L_j^{\cdot}$ is the derivative of L_j and T the symbol for transposition). All elements of the matrices G_i, L_i $(i = 1, \ldots, p + 1)$ are T-periodic analytic functions of $t \in [0, \infty)$.

The set of points $t \in [0, T]$, with the possible exception of a finite set of points M, will be denoted by $[0, T] \setminus M$.

Lemma 1. 1) Each of the functions $F_i(t) = \operatorname{rank} G_i(t)$ $(i = 1, \ldots, p+1)$, considered in the interval $[0, T] \setminus M$, maintains a constant value N_i ($1 \leqslant N_i \leqslant p$; $N_i = 0$ if all elements of G_i vanish identically on $[0, T]$), and for all $t \in [0, T] \setminus M$ the same system of N_i column-vectors of the matrix G_i $(i = 1, \ldots, p + 1)$ is linearly independent.

2) There exists a constant number s (we have in mind the smallest number s, $2 \leqslant s \leqslant p + 1$, with this property) such that for all $t \in [0, T] \setminus M$

$$\operatorname{rank} G_{s-1} = \operatorname{rank} G_s = N \quad (1 \leqslant N = \text{const} \leqslant p) \tag{2.2}$$

Proof. 1) Consider the set $\Delta_i = \{F_{ij}(t)\}$ of all possible square matrices obtained from $G_i(t)$ by deleting columns and rows. The determinants $|F_{ij}(t)|$ are analytic functions and can vanish only on a finite set M of values $t \in [0, T]$, unless they vanish identically for all $t \in [0, T]$ /3/. By definition, the function $F_i(t)$ $(i = 1, \ldots, p + 1)$ equals at each point $t \in [0, T]$ the maximum order $k_i(t)$ of a non-zero deterinant $|F_{ij}|$. Put $k_i^+ = \max k_i(t)$, $t \in [0, T]$. Now, when all elements of $G_i(t)$ vanish identically on $[0, T]$, the statement of the lemma is obvious (in that case $F_i \equiv 0$, $t \in [0, T]$), so we may assume that $1 \leqslant k_i^+ \leqslant p$. Thus, the set Δ_i will contain a matrix F_{ij}^* of dimensions $k_i^+ \times k_i^+$ such that $|F_{ij}^*| \neq 0$ for at least one $t_i = t_{*i} \in [0, T]$. But then, by the properties of analytic functions, $|F_{ij}^*| \neq 0$ for all $t \in [0, T] \setminus M$. Therefore $k_i(t) = k_i^+$ for all $t \in [0, T] \setminus M$. Put $N_i = k_i^+$; then $F_i = N_i$, $t \in [0, T]$. By what we have proved, there exists $t \in t_i^* \in [0, T]$ such that the system of N_i column-vectors of $G_i(t)$ is linearly independent. The elements of these vectors form a square matrix of dimensions $N_i \times N_i$, whose determinant does not vanish at $t = t_i^*$ and hence at any $t \in [0, T] \setminus M$. Thus, this system of N_i column-vectors is linearly independent for all $t \in [0, T] \setminus M$. This proves the first part of the lemma.

2) For all $t \in [0, T]$ there exists $s = s(t)$ such that (2.2) holds. Let $k = \max s(t)$, $t \in [0, T]$. Then, by the first part of the lemma, the number $N^+ = \operatorname{rank} G_R(t^+)$ (t^+ is the value of t at which $\max s(t)$) is attained) remains constant for all $t \in [0, T] \setminus M$. Thus the equality $\operatorname{rank} G_{k-1} = \operatorname{rank} G_k$ is true not only for $t = t^+$ but for all $t \in [0, T] \setminus M$. Putting $s = k$, we see that the proof is complete.

Remark. The assumption about the possible existence of a finite number of exceptional points in $[0, T]$ is essential. This is clear, for example, even for the system $y_1^{\cdot} = \sin t z_1 + z_2$, $z_i^{\cdot} = z_i$ $(i = 1, 2)$, when $\operatorname{rank} G_2 = \operatorname{rank} G_3 = 2$ for all $t \in [0, 2\pi]$ except $t_1 = \pi/2$, $t_2 = 3\pi/2$, whereas at the same time $\operatorname{rank} G_2 \neq \operatorname{rank} G_3$ at $t = t_i$ $(i = 1, 2)$.

3. Construction of the auxiliary system. Let s be the least number such that (2.2) holds for all $t \in [0, T] \setminus M$. By Lemma 1, G_{s-1} contains N columns which are linearly independent at $t \in [0, T] \setminus M$, say $g_i = [g_{i1}(t), \ldots, g_{ip}(t)]$, $i = 1, \ldots, N$.

To construct the auxiliary system, we introduce new variables

$$\mu_i = \sum_{j=1}^{p} g_{ij}(t) z_j \quad (i = 1, \ldots, N) \tag{3.1}$$

Since (2.2) is true, it follows from the structure of the matrices G_i $(i = 1, \ldots, p + 1)$ that this transformation of variables yields a linear system

$$\xi^{\cdot} = Q\xi, \quad \xi = (y_1, \ldots, y_m, \mu_1, \ldots, \mu_N) \tag{3.2}$$

Here Q is an $(m + N) \times (m + N)$ matrix whose elements are functions continuous for all $t \in [0, T] \setminus M$ (and hence also for all $t \in [0, \infty) \setminus M^*$, where M^* is a denumerable set), analytic in the intervals of continuity and T-periodic in $t \in [0, \infty)$. The discontinuities of the elements of Q are due to the possible failure of the vectors g_i to be linearly independent at a finite set of points in $[0, T]$, and hence at a denumerable set of points M^* in $[0, \infty)$. Thus:

Lemma 2. Given system (2.1), one can construct an auxiliary linear system (3.2) of dimensions $m + N$, $N = \operatorname{rank} G_{s-1}(t)$.

4. The structure of the auxiliary system and its solutions. We define several new matrices: 1) $R_1(t)$ is the $N \times p$ matrix whose rows are the columns of G_{s-1} that are linearly independent for $t \in [0, T] \setminus M$. 2) $R_2(t)$ is the $N \times N$ matrix whose columns are the columns of R_1 that are linearly independent for $t \in [0, T] \setminus M$ (we may assume that these are the columns numbered $i_1, \ldots, i_N$ in R_1). 3) $R_3(t)$ is the $p \times N$ matrix whose i_j-th row $(j = 1, \ldots, N)$ is the j-th row of the matrix R_2^{-1}; all the other rows of R_3 vanish for all $t \in [0, T]$ (the elements of the matrix $R_2^{-1}(t)$, and hence also of R_3, may have discontinuities at a finite number of points of $[0, T]$). Finally:

$$R_4 = \begin{Vmatrix} E_m & 0 \\ 0 & R_1 \end{Vmatrix}, \quad R_5 = \begin{Vmatrix} E_m & 0 \\ 0 & R_3 \end{Vmatrix}, \quad R_6 = R_5^{\cdot} \tag{4}$$

Lemma 3. The auxiliary linear system (3.2) may be written as

$$\xi^{\cdot} = R_4 [A^* R_5 - R_6]\xi \tag{4.1}$$

Proof. The transformation from system (2.1) to (3.2) is equivalent to a linear change of variables

$$\mathbf{w} = R(t)\,\mathbf{x}; \quad R = \begin{Vmatrix} E_m & 0 \\ 0 & R^* \end{Vmatrix}, \quad R^* = \begin{Vmatrix} R_1 \\ R_* \end{Vmatrix}$$

where R_* is an arbitrary $(p - N) \times p$ matrix whose elements are analytic T-periodic functions such that R is non-singular for $t \in [0, T] \setminus M$. With this notation, system (3.2) is made up of the first $m + N$ equations of the system $\mathbf{w}^{\cdot} = R[A^*R^{-1} - R^{-1}]\mathbf{w}$. Using the scheme for analysing the structure of matrices of type RA^*R^{-1} described in /4/, one can show that the first $m + N$ rows of this matrix, considered at all $t \in [0, T]$, satisfy the relation $RA^*R^{-1} = R_4A^*R_5$. In addition, a direct check shows that the first $m + N$ rows of the matrix $RR^{\cdot -1}$, considered for all $t \in [0, T]$, satisfy the equality $RR^{\cdot -1} = R_4R_6$. These relations imply that Eqs. (4.1) are indeed those of system (3.2), proving the lemma.

In every interval I_i where the coefficients of system (3.2) are continuous (and therefore analytic), the conditions of the existence and uniqueness theorem of /5/ hold. In each interval I_i therefore, there exists a unique solution $\xi = \xi_{(3.2)}(t)$ of system (3.2) satisfying appropriate initial conditions in the interval and continuable to the entire interval I_i; in addition, these solutions are analytic functions of $t \in I_i$. Any sequence of such solutions (each sequence containing exactly one solution for each interval I_i) may be treated as a solution of system (3.2) for $t \in [0, \infty)$ in the sense that these solutions are defined for almost all $t \in [0, \infty)$ (more precisely: for all $t \in [0, \infty)$ with the exception of a denumerable set M^* of points of discontinuity of the coefficients of system (3.2) in $[0, \infty)$) and satisfy system (3.2) for all $t \in [0, \infty) \setminus M^*$. To find these solutions, it suffices to determine a fundamental matrix $G(t)$ of solutions of system (3.2) in the intervals I_i.

Let us determine the matrix G in the intervals I_i. By Floquet's Theorem /5-7/, and fundamental matrix of solutions of system (2.1) has the form $X = \Phi e^{tL}$, where Φ is an analytic T-periodic matrix-valued function of dimensions $n \times n$, such that $|\Phi| \neq 0$, $t \in [0, \infty)$, L is a constant matrix of the same dimensions. (The matrix-valued function e^{tL} is a fundamental matrix of solutions of the system $\eta^{\cdot} = L\eta$. The roots ρ_i $(i = 1, \ldots, n)$ of the equation $|L - \rho E_n| = 0$ are called the characteristic exponents of system (2.1)). Since system (3.2) is obtained from (2.1) by a linear change of variables (whose coefficients are analytic T-periodic functions of $t \in [0, \infty)$) which is non-singular at $t \in I_i$, and system (3.2) has dimensions $m + N$, it follows that any fundamental matrix of solutions of system (3.2) in I_i has the form $G = \Psi e^{tK}$, where Ψ is an analytic T-periodic matrix-valued function of t, of dimensions $(m + N) \times (m \times N)$, such that $|\Psi| \neq 0$, $t \in I_i$, K is a constant matrix of the same dimensions. The roots ω_j $(j = 1, \ldots, m + N)$ of the equation $|K - \omega E_{m+N}| = 0$ are called the characteristic exponents of system (3.2). The set $\{\omega_1, \ldots, \omega_{m+N}\}$ is a subset of the set of characteristic exponents of system (2.1).

Throughout the sequel, we shall assume without loss of generality that $t = 0$ is not a point of discontinuity of the coefficients of system (3.2); then, since

$$G(0) = \Psi(0), \quad G(T) = \Psi(T)e^{TK}, \quad \Psi(0) = \Psi(T)$$

we conclude that

$$K = [\ln G(T) - \ln G(0)] / T \tag{4.2}$$

As it turns out, in the context of stability analysis with respect to $y_1, \ldots, y_m$ of the unperturbed motion $y = 0, z = 0$ of system (2.1), the above interpretation of a solution of system (3.2) for $t \in [0, \infty]$ is unnecessarily broad. Indeed, the essential purpose of the auxiliary system (3.2) is to investigate the behaviour of the solutions $\mu_i(t)$ $(i = 1, \ldots, N)$ of system (2.1) defined by the variables (3.1). We are therefore interested in those solutions $\xi = \xi_{(3.2)}(t)$ of system (3.2) which, at points $t \in [0, \infty) \setminus M^*$, are also solutions $\xi_{(2.1)}(t) = [y_1(t), \ldots, y_m(t), \mu_1(t), \ldots, \mu_N(t)]$ of system (2.1). Since system (3.2) is obtained from (2.1) in the intervals I_i by a non-singular linear change of variables, the desired set of solutions of system (3.2) exists. Denote this set by E. Note that for all $t_0 \in [0, \infty) \setminus M^*$ and $\xi(t_0)$ there is a uniquely defined solution $\xi_{(3.2)}(t) \in E$.

What we need to know from this point on is not the explicit form of the solutions $\xi_{(3.2)}(t) \in E$ but only the nature of their behaviour at points $t \in [0, \infty) \setminus M^*$ (boundedness and convergence to zero as $t \to \infty$). To this end, we need a rigorous definition of Lyapunov-stability of a motion (solution) $\xi = 0$ of system (3.2). (The solution $\xi = 0$ of system (3.2) is understood in the sense that for all $t \in [0, \infty) \setminus M^*$ it exists - since system (3.2) satisfies the conditions of the existence and uniqueness theorem - and satisfies the system for $t \in [0, \infty) \setminus M^*$).

Definition 1. A motion $\xi = 0$ of system (3.2) is said to be Lyapunov-stable if, for any $\varepsilon, t_0 \geqslant 0$ $(t_0 \in [0, \infty) \setminus M^*)$, there exists $\delta(\varepsilon, t_0) > 0$ such that, if $\|\xi(t_0)\| < \delta$, then $\|\xi_{(3.2)}(t; t_0, \xi(t_0))\| < \varepsilon$, $\xi_{(3.2)}(t) \in E$ for all $t \in [0, \infty) \setminus M^*$. If moreover $\lim \|\xi_{(3.2)}(t; t_0, \xi(t_0))\| = 0$, $t \to \infty$, then the motion $\xi = 0$ is asymptotically Lyapunov-stable.

In view of the structure of the fundamental matrix of solutions $G(t)$ for system (2.1) and thanks to (4.2), we obtain

Lemma 4. A motion $\xi = 0$ of system (3.2) is asymptotically Lyapunov-stable if and only if all the roots of the equation

$$|[\ln G(T) - \ln G(0)] / T - \omega E_{m+N}| = 0 \tag{4.3}$$

have negative real parts.

5. A criterion for the partial stability of linear systems with periodic analytic coefficients.

Theorem 1. A necessary and sufficient condition for asymptotic stability with respect to $y_1, \ldots, y_m$ of the unperturbed solution $\mathbf{y} = 0, \mathbf{z} = 0$ of system (2.1) is that the motion $\xi = 0$ of system (3.2) by asymptotically Lyapunov-stable, i.e., all the roots of Eq.(4.3) have negative real parts.

Proof. The structure of the fundamental matrix of solutions $X(t)$ of system (2.1) is such that asymptotic stability with respect to $y_1, \ldots, y_m$ of the motion $\mathbf{y} = 0, \mathbf{z} = 0$ of the system is precisely exponential asymptotic stability. Direct integration of the first m equations of system (2.1) will show that a necessary condition for the latter is that, on trajectories of system (2.1),

$$|\mu_i(t)| = \Big|\sum_{l=1}^{p} b_{il}(t) z_l(t)\Big| \leqslant \alpha_i \exp[-\beta_i(t - t_0)] \quad (i = 1, \ldots, m)$$

where α_i, β_i are positive constants. Therefore, to establish asymptotic stability relative to $y_1, \ldots, y_m$ of the motion $\mathbf{y} = 0, \mathbf{z} = 0$ of system (2.1), we must show that it is also asymptotically stable with respect to the variables μ_i $(i = 1, \ldots, m)$.

Consider those of the variables μ_i $(i = 1, \ldots, m)$ that are linearly independent for $t \in [0, T] \setminus M$ (suppose these are $\mu_1, \ldots, \mu_{m_1}$, $m_1 \leqslant m$) and put $\xi = (y_1, \ldots, y_m, \mu_1, \ldots, \mu_{m_1})$. Then there are two possibilities: (1) an auxiliary system of type (3.2) can be constructed; (2) construction of the auxiliary system for this specific set of variables in the vector ξ is impossible. Now, in case (1) the unperturbed motion $\mathbf{y}=0, \mathbf{z}=0$ of system (2.1) will be asymptotically stable with respect to y_i, μ_j $(i=1,\ldots,m;\ j=1,\ldots,m_1)$ if and only if the motion $\xi=0$ of system (3.2) is asymptotically Lyapunov-stable; hence, in that case, the proof that asymptotic Lyapunov-stability of $\xi=0$ is indeed necessary is complete. Since the reduction to a system of type (3.2) is in the final analysis always possible, by introducing supplementary variables in the vector ξ, it follows again, as in case (1), that the necessity of asymptotic Lyapunov-stability of the motion $\xi=0$ is proved. By Lemma 4, the motion $\xi=0$ of system (3.2) is asymptotically Lyapunov-stable if and only if all roots of Eq.(4.3) have negative real parts; hence the necessity part of the proof is complete. The sufficiency is obvious.

Corollary 1. A necessary and sufficient condition for (non-asymptotic) stability with respect to $y_1,\ldots,y_m$ of the unperturbed solution $\mathbf{y}=0, \mathbf{z}=0$ of system (2.1) is that the motion $\xi=0$ of system (3.2) be Lyapunov-stable, i.e., all roots of Eq.(4.3) either have negative real parts, are zero or pure imaginary, and in the latter two cases the elementary divisors corresponding to multiple roots are simple.

Remarks. 1°. Since our stability criteria are inequalities involving the roots of Eq. (4.3), these roots may be determined using numerical and approximate methods for determining the fundamental matrices $G(0)$, $G(T)$ of solutions of the auxiliary system (3.2). Thus, the problem of investigating partial stability in an infinite time interval $[0,\infty)$ reduces to numerical integration in a finite interval $[0, T]$.

2°. Suppose that condition (2.2) holds and the matrix G_{s-1}, for all $t\in[0,\infty)$ may be expressed as

$$G_{s-1}=\{g_{ij}\},\ g_{ij}=g_j g_{ij}^*\ \ (i=1,\ldots,\ m+N,\ j=1,\ldots,\ p)$$

where g_j, g_{ij}^* are analytic T-periodic functions of $t\in[0,\infty)$ such that $\operatorname{rank} G_{s-1}^*=\operatorname{rank} G_s^*=N$, $G_s^*=G_{s-1}^*+D^T G_{s-1}^*$, $G_{s-1}^*=\{g_{ij}^*\}$, $2\leqslant s\leqslant p+1$, $t\in[0,T]$. Then, by transforming system (2.1) to new variables

$$\mu_i=\sum_{j=1}^{p} g_{ij}^*(t)z_j\quad (i=1,\ldots,N)$$

one can construct an auxiliary linear system of type (3.2) with analytic T-periodic coefficients.

Example. Consider the following instance of system (2.1):

$$y_1^{\cdot}=-y_1+\sin 2tz_1+2\cos^2 tz_2 \tag{5.1}$$

$$z_1^{\cdot}=-\cos ty_1-z_1+z_2,\ \ z_2^{\cdot}=\sin ty_1-z_1-z_2$$

The auxiliary linear system (3.2) is then

$$y_1^{\cdot}=-y_1+\mu_1,\ \ \mu_1^{\cdot}=(-1-\operatorname{tg} t)\,\mu_1 \tag{5.2}$$

$$(\mu_1=\sin 2tz_1+2\cos^2 tz_2)$$

Integrating system (5.2), we determine its solutions

$$y_1(t)=\{[y_1(t_0)-\operatorname{tg}t_0\mu_1(t_0)]+\mu_1(t_0)\sin t\,/\cos t_0\}\,e^{-(t-t_0)}$$

$$\mu_1(t)=[\mu_1(t_0)\cos t/\cos t_0]\,e^{-(t-t_0)},\ \ t\geqslant t_0\geqslant 0$$

Since the characteristic exponents $\omega_1=\omega_2=-1$ of system (5.2) have negative real parts. the unperturbed motion $y_1=z_1=z_2=0$ of system (5.1) is asymptotically y_1-stable (Theorem 1).

Introducing the new variable $\mu_2=\sin tz_1+\cos tz_2$, we can also consider the system

$$y_1^{\cdot}=-y_1+2\cos t\mu_2,\ \ \mu_2^{\cdot}=-\mu_2 \tag{5.3}$$

which has analytic coefficients. Systems (5.2) and (5.3) have the same characteristic exponents.

6. Partial stability in the linear approximation.

Let the equations of perturbed motion be

$$y_i^{\cdot}=\sum_{k=1}^{m} a_{ik}y_k+\sum_{l=1}^{p} b_{il}z_l+Y_i(t,\mathbf{y},\mathbf{z}) \tag{6.1}$$

$$z_j^{\cdot}=\sum_{l=1}^{p} d_{jl}z_l+Z_j(t,\mathbf{y},\mathbf{z})\quad (i=1,\ldots,m;\ j=1,\ldots,p)$$

Here a_{ik}, b_{il}, d_{jl} are constants, Y_i, Z_j non-linear terms. We assume that the right-hand sides of system (6.1) are continuous in the domain

$$t\geqslant 0,\ \|\mathbf{y}\|<H,\ \|\mathbf{z}\|<\infty \tag{6.2}$$

and satisfy the uniqueness conditions there, and that the solutions of system (6.1) are z-continuable /2/.

Express the functions $Y_i\ (i = 1, \ldots, m)$ as

$$Y_i(t, \mathbf{y}, \mathbf{z}) = Y_i^\circ(\mathbf{z}) + \sum_{j=1}^{N} Y_j^*(\mathbf{y})\,\bar{Y}_{ij}^\circ(\mathbf{z}) + Y_i^{**}(t, \mathbf{y}, \mathbf{z}) \tag{6.3}$$

where $Y_i^\circ, \bar{Y}_{ij}^\circ$ are functions defined by

$$Y_i^\circ(\mathbf{z}) = \sum_{v=2}^{r} U_v^{(i)}(\mathbf{z}), \quad \bar{Y}^\circ_{ij}(\mathbf{z}) = \sum_{v=1}^{s} U_v^{(ij)}(\mathbf{z}) \tag{6.4}$$

in which $U_l^{(i)}, \bar{U}_v^{(ij)}$ are homogeneous forms in the variables $z_1, \ldots, z_p$, of degrees $l\ (l \leqslant r)$ and $v\ (v \leqslant s)$, respectively, where r, s are finite numbers. The functions $Y_j^*(\mathbf{y})$ are analytic in the domain $\|y\| < H$, and $Y_j^*(0) = 0$.

The essential purpose of the expansions (6.3) and (6.4) is to extract from $Y_i\ (i = 1, \ldots, m)$ those terms $Y_i^\circ, \bar{Y}_{ij}^\circ\ (i = 1, \ldots, m;\ j = 1, \ldots, N)$ which will be used as supplementary variables to form the auxiliary linear system, while the remaining terms $Y_i^{**}\ (i = 1, \ldots, m)$ will be estimated (from the standpoint of stability relative to $y_1, \ldots, y_m$) in terms of y_i, Y_i°, $\bar{Y}_{ij}^\circ\ (i = 1, \ldots, m;\ j = 1, \ldots, N)$. Such expansions of the functions $Y_i\ (i = 1, \ldots, m)$ are not unique, as one can include in $Y_i^\circ, \bar{Y}_{ij}^\circ$ various sets of terms of the indicated type; this arbitrary element should be exploited in order to rationalize the search for the most acceptable solution.

As a first-approximation system for (6.1) we take the equations

$$y_i^{\cdot} = \sum_{k=1}^{m} a_{ik} y_k + \sum_{l=1}^{p} b_{il} z_l + Y_i^\circ(\mathbf{z}) + \sum_{j=1}^{N} Y_j^*(\mathbf{y})\,\bar{Y}_{ij}^\circ(\mathbf{z}) \tag{6.5}$$

$$z_j^{\cdot} = \sum_{l=1}^{p} d_{jl} z_l \quad (i = 1, \ldots, m;\ j = 1, \ldots, p)$$

We shall show that, when investigating stability relative to $y_1, \ldots, y_m$ of the unperturbed motion $\mathbf{y} = 0$, $\mathbf{z} = 0$ of system (6.1), one can apply non-linear transformations to replace system (6.5) by a specially constructed linear system. Indeed, define new variables by

$$\mu_i^{(1)} = \sum_{l=1}^{p} b_{il} z_l + Y_i^\circ(\mathbf{z}) = \sum_{l=1}^{p} b_{il} z_l + \sum_{v=2}^{r} U_v^{(i)}(\mathbf{z}) \tag{6.6}$$

$$\mu_{ij}^{(2)} = \bar{Y}_{ij}^\circ(\mathbf{z}) = \sum_{v=1}^{s} U_v^{(ij)}(\mathbf{z}) \quad (i, j = 1, \ldots, m)$$

When this is done, there are two possibilities.

Case 1. System (6.5) becomes (we assume without loss of generality that $N = m$)

$$y_i^{\cdot} = \sum_{k=1}^{m} a_{ik} y_k + \mu_i^{(1)} + \sum_{j=1}^{m} Y_j^*(\mathbf{y})\,\mu_{ij}^{2} \tag{6.7}$$

$$\mu_j^{\cdot(1)} = \sum_{v=1}^{r} U_v^{(j)*}(z) = \sum_{l=1}^{m} L_{jl}^{(1)} \mu_l^{(1)} + \sum_{l,\varepsilon=1}^{m} \bar{L}_{jl\varepsilon}^{(1)} \mu_{l\varepsilon}^{(2)}$$

$$\mu_{\gamma\theta}^{\cdot(2)} = \sum_{v=1}^{s} U_v^{(\gamma\theta)*}(z) = \sum_{l=1}^{m} L_{\gamma\theta l}^{(2)} \mu_l^{(1)} + \sum_{l,\varepsilon=1}^{m} \bar{L}_{\gamma\theta l\varepsilon}^{(2)} \mu_{l\varepsilon}^{(2)}$$

$$z_\varepsilon^{\cdot} = \sum_{l=1}^{p} d_{\varepsilon l} z_l \quad (i, j, \gamma, \theta = 1, \ldots, m;\quad \varepsilon = 1, \ldots, p) \tag{6.8}$$

where $U_v^{(j)*}$, $U_v^{(\gamma\theta)*}$ are homogeneous forms in $z_1, \ldots, z_p$ of order v; $L_{jl}^{(1)}, \bar{L}_{jl\varepsilon}^{(1)}, L_{\gamma\theta l}^{(2)}, \bar{L}_{\gamma\theta l\varepsilon}^{(2)}$ are constants. The behaviour of the variables describing the state of system (6.7) is completely determined by the behaviour of the variables $y_1, \ldots, y_m$ of system (6.5).

Case 2. Let us assume that all the equalities in the second and third groups of Eqs. (6.7) fail to hold. Then we again define new variables

$$\bar{\mu}_j^{(1)} = \sum_{v=2}^{r} U_v^{(j)*}(\mathbf{z}), \quad \bar{\mu}_{\gamma\theta}^{(2)} = \sum_{v=1}^{s} U_v^{(\gamma\theta)*}(\mathbf{z}) \quad (j, \gamma, \theta = 1, \ldots, m)$$

Associating with the new variables vectors that characterize them (see /8/), one can show that, by continuing if necessary to introduce new variables, one can always transform from system (6.5) to a finite-dimensional system of type (6.7).

In the course of this transformation, the original system (6.1) is reduced to the form (6.7), (6.8), with the right-hand sides of the equations augmented by the addition of the respective terms

$$Y_i^{**}(t, \mathbf{y}, \mathbf{z}),\ Z_j^{(1)}(t, \mathbf{y}, \mathbf{z}) = \sum_{l=1}^{p} (\partial\mu_j^{(1)}/\partial z_l) Z_l \tag{6.9}$$

$$Z_{\gamma\theta}^{(2)}(t, \mathbf{y}, \mathbf{z}) = \sum_{l=1}^{p} (\partial\mu_{\gamma\theta}^{(2)}/\partial z_l) Z_l,\ Z_\varepsilon(t, y, z) \quad (i, j, \gamma, \theta = 1, \ldots, m)$$

In the general case, when construction of a system of type (6.7) is possible at some finite stage of the procedure, system (6.1) is transformed into a system of type (6.7), (6.8) with added terms (6.9); the linear part of this system (excluding the last group of equations) forms a closed linear steady-state system with respect to y_i $(i = 1, \ldots, m)$ and the additional terms form the auxiliary linear steady-state system.

When investigating asymptotic stability with respect to $y_1, \ldots, y_m$ of the unperturbed motion $\mathbf{y} = 0$, $\mathbf{z} = 0$ of system (6.1), the above transformation from the non-linear system (6.5) to a system of type (6.7) enables one to replace the linear-approximation equations

$$y_i^\cdot = \sum_{k=1}^{m} a_{ik} y_k + \sum_{l=1}^{p} b_{il} z_l, \quad z_j^\cdot = \sum_{l=1}^{p} d_{jl} z_l$$
$$(i = 1, \ldots, m;\ j = 1, \ldots, p)$$

with a specially constructed linear-approximation system - the linear part of system (6.7), which is equivalent (in the context of stability relative to $y_1, \ldots, y_m$) to the non-linear approximation (6.5) of the initial non-linear system (6.1). When this is done, the linear-approximation equations for system (6.1) include part of its non-linear terms and permits a simpler derivation of the necessary estimates for the remaining group of non-linear terms. This approach enables us to enlarge the class of non-linear systems for which the problem of partial stability is solvable by a linear approximation /9, 10/.

Let ξ_i $(i = 1, \ldots, m^2 + 2m)$ denote the components of the vector ξ consisting of the variables y_i, $\mu_j^{(1)}$, $\mu_{\gamma\theta}^{(2)}$ $(i, j, \gamma, \theta = 1, \ldots, m)$ that determine the state of system (6.7), and Y_{*i} $(i = 1, \ldots, m^2 + 2m)$ the components of the vector-function Y_* consisting of the functions Y_i^{**}, $Z_j^{(1)}$, $Z_{\gamma\theta}^{(2)}$ $(i, j, \gamma, \theta = 1, \ldots, m)$.

Assume that in the domain $t \geqslant 0$, $\|\xi\| < H$, $\|\mathbf{z}\| < \infty$ one has the condition

$$|Y_*(t, \mathbf{y}, \mathbf{z})| \leqslant \alpha \|\xi\| \tag{6.10}$$

where α, H are sufficiently small positive constants.

Theorem 2. Let all the roots of the characteristic equation of the linear part of system (6.7) have negative real parts. Then the unperturbed motion $\mathbf{y} = 0, \mathbf{z} = 0$ of system (6.1) is asymptotically stable with respect to $y_1, \ldots, y_m$ of its non-linear terms satisfy condition (6.10).

Proof. Under the assumptions of the theorem, with the notation introduced above for the linear part of system (6.7), one can find a function $V = V(\xi)$ satisfying the conditions of Lyapunov's Theorem:

$$c_1 \|\xi\|^2 \leqslant V \leqslant c_2 \|\xi\|^2, \quad V^\cdot_{(6.8)} \leqslant -c_3 \|\xi\|^2 \tag{6.11}$$

where $V^\cdot_{(6.7)}$ is the derivative of V along trajectories of the linear part of system (6.7), c_i $(i = 1, 2, 3)$ are positive constants. Now, differentiating V along trajectories of system (6.1):

$$V^\cdot_{(6.1)} = V^\cdot_{(6.7)} + \sum_{s=1}^{m} \frac{\partial V}{\partial \xi_s} \sum_{\gamma=1}^{m} Y_\gamma^*(\mathbf{y}) \mu_{s\gamma} + \sum_{j=1}^{m^2+2m} \frac{\partial V}{\partial \xi_j} Y_{*j}(t, \mathbf{y}, \mathbf{z}) \quad (N = m)$$

and taking into account that, by (6.10), (6.11), $Y_\gamma^*(0) = 0$ $(\gamma = 1, \ldots, m)$ in the domain $t \geqslant 0$, $\|\xi\| < H$, $\|\mathbf{z}\| < \infty$ we obtain an estimate $V^\cdot_{(6.1)} \leqslant -c_3 \|\xi\|^2 + \beta \|\xi\|^2$, where β is a sufficiently small constant. Hence there exists a number $c > 0$ such that $V^\cdot_{(6.1)} \leqslant c \|\xi\|^2$. Thus the function V satisfies the conditions of the theorem on partial asymptotic stability /2/ and the unperturbed motion $\mathbf{y} = 0, \mathbf{z} = 0$ of system (6.1) is asymptotically ξ-stable. Since $\xi_i = y_i$ $(i = 1, \ldots, m)$, this completes the proof.

Example. Consider the following system (6.1):

$$
\begin{aligned}
&y_1^{\cdot} = -y_1 + Y_1^* (y_1) z_1 z_2 + z_2^2 z_3 + Y_1^{**} (t, y_1, z_1, z_2, z_3) \\
&z_j^{\cdot} = \Sigma_j + Z_j (t, y_1, z_1, z_2, z_3) \quad (j = 1, 2, 3) \\
&(\Sigma_1 = z_1, \ \Sigma_2 = -2z_2, \ \Sigma_3 = z_2 + 2z_3)
\end{aligned}
\tag{6.12}
$$

where Y_1^{**}, Z_j are analytic functions in the domain (6.2) with continuous bounded coefficients.

Among the roots of the characteristic equation of the linear part of system (6.12), there are some with positive real parts, and so the motion $y_1 = z_1 = z_2 = 0$ of the system is Lyapunov-unstable. Let us see what happens with regard to asymptotic y_1-stability. Taking equations of type (6.5) to be the first-approximation system and defining new variables $\mu_1 = z_1 z_2$, $\mu_2 = z_2^2 z_3$, we construct the system

$$
\begin{aligned}
&y_1^{\cdot} = -y_1 + Y_1^* (y_1) \mu_1 + \mu_2 + Y_1^{**} \quad (t, y_1, z_1, z_2, z_3) \\
&\mu_i^{\cdot} = \Sigma_i^* + Z_{3+i} \quad (t, y_1, z_1, z_2, z_3) \\
&z_j^{\cdot} = \Sigma_j + Z_j \quad (t, y_1, z_1, z_2, z_3) \quad (i, j = 1, 2, 3) \\
&(\Sigma_1^* = -\mu_1, \ \Sigma_2^* = \mu_3, \ \Sigma_3^* = -6\mu_2 - 7\mu_3 \\
&Z_4 = z_1 Z_2 + z_2 Z_1, \ \ Z_5 = 2z_2 z_3 Z_2 + z_2^2 Z_3, \ \ Z_6 = -Z_3 + 3z_2^2 Z_2)
\end{aligned}
\tag{6.13}
$$

Assume that in the domain $t \geqslant 0$, $|y_1| < H$, $\|\boldsymbol{\mu}\| < H$, $\|\mathbf{z}\| < \infty$ the following conditions hold ($Y_* = (Y_1^{**}, Z_4, Z_5, Z_6)$ and the summation over j is from 1 to 3):

$$
| Y_* (t, y_1, z_1, z_2, z_3) | \leqslant \alpha_0 | y_1 | + \Sigma \alpha_j | \mu_j |
\tag{6.14}
$$

where α_s $(s = 0, \ldots, 3)$ are sufficiently small positive constants. Since all the roots of the characteristic equation of the linear part of the first four equations in (6.13) have negative real parts. It follows by Theorem 2 that, provided condition (6.14) holds, the motion $y_1 = z_1 = z_2 = z_3 = 0$ of system (6.12) is asymptotically y_1-stable.

7. Additional possibilities for investigating partial stability in the linear approximation. We now consider a more general choice of new variables in the construction of the auxiliary linear-approximation system for system (6.1). Express Y_i $(i = 1, \ldots, m)$ in the following form (throughout this section, the ranges of summation are: with respect to v - from 2 to r, with respect to k - from 1 to m, and with respect to l - from 1 to p):

$$
Y_i (t, \mathbf{y}, \mathbf{z}) = Y_i^{\circ} (\mathbf{y}, \mathbf{z}) + Y_i^{**} (t, \mathbf{y}, \mathbf{z}), \quad Y_i^{\circ} = \Sigma U_v^{(i)} (\mathbf{y}, \mathbf{z})
\tag{7.1}
$$

where $U_\gamma^{(i)}$ are homogeneous forms of finite degree γ in $\mathbf{y}, \mathbf{z}$ and define new variables by

$$
\mu_i = \Sigma b_{il} z_l + \Sigma U_v^{(i)} (\mathbf{y}, \mathbf{z}) \quad (i = 1, \ldots, m)
\tag{7.2}
$$

When this is done, there are two possibilities. In the first case, system (6.1) becomes

$$
\begin{aligned}
&y_i^{\cdot} = \Sigma a_{ik} y_k + \mu_i + Y_i^{**} (t, \mathbf{y}, \mathbf{z}) \\
&\mu_j^{\cdot} = \Sigma b_{jl}^* z_l + \Sigma U_v^{(j)*} (\mathbf{y}, \mathbf{z}) + Z_j^* (t, \mathbf{y}, \mathbf{z}) = \Sigma e_{jk} \mu_k + Z_j^* (t, \mathbf{y}, \mathbf{z}) \\
&z_s^{\cdot} = \Sigma d_{se} z_l + Z_s (t, \mathbf{y}, \mathbf{z}) \\
&Z_j^* = \Sigma \, (\partial Y_j^{\circ} / \partial y_k) Y_k + \Sigma \, (\partial Y_j^{\circ} / \partial z_l) Z_l \\
&(i, j = 1, \ldots, m; \ s = 1, \ldots, p)
\end{aligned}
\tag{7.3}
$$

where b_{jl}^*, e_{jk} are constants and $U_\gamma^{(j)*}$ are homogeneous forms in $\mathbf{y}, \mathbf{z}$ of order γ. One can then extract from (7.3) a linear steady-state system

$$
\begin{aligned}
&y_i^{\cdot} = \Sigma a_{ik} y_k + \mu_i, \ \ \mu_j^{\cdot} = \Sigma e_{jk} \mu_k \\
&(i, j = 1, \ldots, m)
\end{aligned}
\tag{7.4}
$$

which will serve as a linear approximation for the first two groups of equations in (7.3).

In the second case, when introduction of the new variables (7.2) does not produce a system of type (7.3), one can show that, by again defining new variables and continuing the procedure if necessary, system (6.1) may always be transformed to a system with a structure of (7.3), from which one can extract a linear steady-state system of type (7.4). This conclusion also holds when the second group of equations in the linear part of system (6.1) is allowed to contain terms that are linear (with constant coefficients) in $y_1, \ldots, y_m$; in that case, instead of (7.4), one can extract from (7.3) a linear system

$$
y_i^{\cdot} = \Sigma a_{ik} y_k + \mu_i, \quad \mu_j^{\cdot} = \Sigma a_{jk}^* y_k + \Sigma e_{jk} \mu_k \quad (i, j = 1, \ldots, m)
\tag{7.5}
$$

Suppose that all the roots of the characteristic equation of system (7.4) (system (7.5)) have negative real parts. Then the unperturbed motion of system (6.1) will be asymptotically stable with respect to $y_1, \ldots, y_m$ if its non-linear terms $Y_* = (Y_1^{**}, \ldots, Y_m^{**}, Z_1^*, \ldots, Z_m^*)$ satisfy conditions (6.10) in the domain $t \geqslant 0$, $\|\boldsymbol{\xi}\| < H$, $\|\mathbf{z}\| < \infty$, $\boldsymbol{\xi} = (y_1, \ldots, y_m, \mu_1, \ldots, \mu_m)$. The proof follows the same lines as that of Theorem 2.

If estimates of type (6.10) cannot be established, similar estimates may be obtained by employing the following devices.

1°. Express each of the expressions y^0_i as a sequence of several new variables (rather than a single variable μ_i) and the variables $y_1, \ldots, y_m$. Here, again, one can always form a system with the structure of (7.3), with a linear part in the form of (7.4) or (7.5). This extends the range of cases for which estimates of type (6.10) will hold.

2°. The procedure of defining new variables may be continued by extracting from Y_i^{**}, Z_j^* $(i, j = 1, \ldots, m)$ some set of non-linear terms and taking these to be the new variables (until a satisfactory solution is obtained). This again yields a system with the structure of (7.3) and increases the possibility of establishing estimates of type (6.10).

For example, the definition of a new variable $\mu_1 = y_1^2 z_1$ does not convert the equation

$$y_1^\cdot = -3y_1 + y_1^2 z_1, \quad z_1^\cdot = 2y_1 + z_1 \tag{7.6}$$

to a closed system with respect to y_1, μ_1, since the equation $\mu_1^\cdot = -5\mu_1 + 2y_1^3 + 2y_1^3 z_1^2$ includes the term $2y_1^3 z_1^2$, which cannot be expressed (as required) in terms of y_1, μ_1. However, by using devices 1°, 2° one can form the following auxiliary systems:

$$1)\ y_1^\cdot = -3y_1 + y_1\mu_1, \quad \mu_1^\cdot = -2\mu_1 + 2y_1^2 + \mu_1^2 \quad (\mu_1 = y_1 z_1);$$

$$2)\ y_1^\cdot = -3y_1 + \mu_1, \quad \mu_1^\cdot = -5\mu_1 + 2y_1^3 + 2\mu_1\mu_2, \quad \mu_2^\cdot = -2\mu_2 + 2y_1^2 + \mu_2^2$$

$$(\mu_1 = y_1^2 z_1, \quad \mu_2 = y_1 z_1)$$

In either case, the null solution of the auxiliary system is asymptotically Lyapunov-stable; consequently, the motion $y_1 = z_1 = 0$ of system (7.6), though Lyapunov-unstable, is asymptotically y_1-stable.

Remark. This approach to the investigation of y-stability in the linear approximation extends the approach proposed in /8/; however, unlike /8/, it is not assumed here that the solutions of the original system of differential equations are bounded and a larger class of non-linearities is considered.

Example. Let the equations of perturbed motion, in which the non-linear perturbations are assumed to be analytic functions in the domain (6.2) with continuous bounded coefficients, be

$$\begin{aligned} y_1^\cdot &= ay_1 + by_1^k (y_1 z_1)^r + Y_1^{**}(t, y_1, z_1) \\ z_1^\cdot &= cy_1 + dz_1 + Z_1(t, y_1, z_1) \end{aligned} \tag{7.7}$$

where a, b, c, d are constants, k, r integers with $k \geqslant 2$, $r \geqslant 1$, $k > r$. If $d > 0$ the motion $y_1 = z_1 = 0$ of system (7.7) is Lyapunov-unstable.

Let us examine this motion for asymptotic y_1-stability. To that end, we define a new variable $\mu_1 = y_1 z_1$ and form the system

$$\begin{aligned} y_1^\cdot &= ay_1 + by_1^k \mu_1^r + Y_1^{**}(t, y_1, z_1) \\ \mu_1^\cdot &= (a + d)\mu_1 + Z_1^*(t, y_1, z_1) \\ z_1^\cdot &= cy_1 + dz_1 + Z_1(t, y_1, z_1) \\ Z_1^* &= cy_1^2 + by_1^{k-1}\mu_1^{r+1} + Z_1^{**}, \quad Z_1^{**} = y_1 Z_1 + z_1 Y_1^{**} \end{aligned}$$

If the following estimate holds in the domain $t \geqslant 0$, $|y_1| < H$, $|\mu_1| < H$, $|z_1| < \infty$ (with $(Y_* = (Y_1^{**}, Z_1^{**}))$

$$|Y_*(t, y_1, z_1)| \leqslant \alpha |y_1| + \beta |y_1 z_1|$$

where α, β are sufficiently small positive constants, with $a < 0$, $a + d < 0$ and c, b arbitrary, then the motion $y_1 = z_1 = 0$ of system (7.7) is asymptotically y_1-stable.

8. Stability in Lyapunov-critical cases.

Let the equations of perturbed motion (in vector notation) be

$$\mathbf{y}^\cdot = A\mathbf{y} + B\mathbf{z} + Y(t, \mathbf{y}, \mathbf{z}), \quad \mathbf{z}^\cdot = C\mathbf{y} + D\mathbf{z} + Z(t, \mathbf{y}, \mathbf{z}) \tag{8.1}$$

$$(\mathbf{x}^\cdot = A^*\mathbf{x} + X(t, \mathbf{x}), \quad \mathbf{x} = (\mathbf{y}, \mathbf{z}))$$

where A, B, C, D are constant matrices of the appropriate dimensions, and the non-linear perturbations Y, Z satisfy the following conditions in the domain $t \geqslant 0$, $\|x < H$ /11, 12/:

$$Y(t, 0, 0) \equiv Y(t, 0, \mathbf{z}) \equiv 0, \quad Z(t, 0, 0) \equiv Z(t, 0, \mathbf{z}) \equiv 0 \tag{8.2}$$

$$(\|Y(t, \mathbf{y}, \mathbf{z})\| + \|Z(t, \mathbf{y}, \mathbf{z})\|)/\|\mathbf{y}\| \rightrightarrows 0 \quad \text{as} \quad \|\mathbf{y}\| + \|\mathbf{z}\| \to 0 \tag{8.3}$$

Consider the more general system

$$\begin{aligned} \mathbf{y}^\cdot &= A\mathbf{y} + B\mathbf{z} + Y^\circ(\mathbf{z}) + Y(t, \mathbf{y}, \mathbf{z}) \\ \mathbf{z}^\cdot &= C\mathbf{y} + D\mathbf{z} + Z(t, \mathbf{y}, \mathbf{z}) \end{aligned} \tag{8.4}$$

where the components $Y_i^\circ\ (i = 1, \ldots, m)$ of the vector-function $Y^\circ(\mathbf{z})$ satisfy the conditions listed in Sect.6.

Theorem 3. Let the motion $\mathbf{y} = 0, \mathbf{z} = 0$ of the system

$$\mathbf{y}^\cdot = A\mathbf{y} + B\mathbf{z}, \quad \mathbf{z}^\cdot = C\mathbf{y} + D\mathbf{z} \tag{8.5}$$

be exponentially asymptotically y-stable and Lyapunov-stable. Then the same is true of the following motions: 1) $\mathbf{y} = 0, \mathbf{z} = 0$ of system (8.1); 2) $\mathbf{y} = 0, \mathbf{z} = 0$ of system (8.4), provided that in addition B is the zero matrix and the null solution of the system

$$\mathbf{y}^\cdot = A\mathbf{y} + Y^\circ(\mathbf{z}), \quad \mathbf{z}^\cdot = D\mathbf{z} \tag{8.6}$$

is exponentially asymptotically y-stable.

Proof. This theorem generalizes well-known theorems of /11, 12/, in the sense that in /11/ B, C and in /12/ B are assumed to be zero matrices, and in both cases $Y^\circ(\mathbf{z}) = 0$. We shall show that systems (8.1) and (8.4) can be modified by the procedures developed in this paper in such a way that the proofs of /11, 12/ are applicable.

1) Transforming the linear part of system (8.1) by the linear change of variables indicated in /4/, we obtain

$$\begin{aligned} \xi^\cdot &= Q_4 A^* Q_5 \xi + Y^*(t, \xi, \eta) \\ \eta^\cdot &= C_1\xi + C_2\eta + Z^*(t, \xi, \eta) \end{aligned} \tag{8.7}$$

where Q_4, Q_5, C_1, C_2 are constant matrices (Q_4, Q_5 are defined in /4/), and the motion $\xi = 0$, $\eta = 0$ of system (8.7) is exponentially asymptotically ξ-stable and Lyapunov-stable. The components of the vector-function Y^*, Z^* are either those of Y, Z or linear combinations thereof; one can therefore show that system (8.7) satisfies the assumptions of /11, 12/ and the solution $\xi = 0$, $\eta = 0$ of system (8.7) is Lyapunov-stable and exponentially asymptotically ξ-stable. Thus the unperturbed motion of system (8.1) is Lyapunov-stable and exponentially asymptotically y-stable.

2) Using the procedure of Sect.6, one can go over from system (8.6) (we may assume without loss of generality that the required auxiliary system is obtained in one step of the procedure) to a linear steady-state system $\mathbf{y}^\cdot = A\mathbf{y} + \mu$, $\mu^\cdot = D^*\mu$, whose null solution $y = 0$, $\mu = 0$ is exponentially asymptotically stable (relative to all the variables). When this is done the intial system (8.4) becomes

$$\begin{aligned} &\mathbf{y}^\cdot = A\mathbf{y} + \mu + (Y(t, \mathbf{y}, \mathbf{z}), \quad \mu^\cdot = D^*\mu + Y^*(t, \mathbf{y}, \mathbf{z}) \\ &\mathbf{z}^\cdot = C\mathbf{y} + D\mathbf{z} + Z(t, \mathbf{y}, \mathbf{z}) \\ &Y^* = (\partial Y^\circ/\partial \mathbf{z})(C\mathbf{y} + Z), \quad \xi = (\mathbf{y}, \mu) \end{aligned} \tag{8.8}$$

and the solution $\xi = 0$, $\mathbf{z} = 0$ of the linear-approximation system for (8.8) is Lyapunov-stable and exponentially ξ-stable. Since

$$\| Y^* \| \leqslant \| C \| \| \partial Y^\circ/\partial \mathbf{z} \| \| y \| + \| \partial Y^\circ/\partial \mathbf{z} \| \| Z \|$$

it follows from (8.3) that, as $\| y \| + \| \mathbf{z} \| \to 0$,

$$(\| Y(t, \mathbf{y}, \mathbf{z})\| + \| Y^*(t, \mathbf{y}, \mathbf{z})\| + \| Z(t, \mathbf{y}, \mathbf{z})\|)/\| \xi \| \rightrightarrows 0$$

and, in addition, by (8.2), $Y^*(t, 0, \mathbf{z}) \equiv 0$. Hence system (8.8) satisfies the conditions of /11, 12/ and its motion $\xi = 0$, $\mathbf{z} = 0$ is Lyapunov-stable and exponentially asymptotically ξ-stable. Thus the unperturbed motion $\mathbf{y} = 0$, $\mathbf{z} = 0$ of system (8.4) is Lyapunov-stable and exponentially asymptotically y-stable.

Remark. the important point in Theorem 3 is not only the exponential y-stability of the motion $\mathbf{y} = 0, \mathbf{z} = 0$ of systems (8.5) and (8.6), but also the form of the variables comprising systems (8.7) and (8.8), respectively. In this case, for example, conditions (8.2) in the first part of the theorem may be replaced by the weaker conditions $Y^*(t, 0, \eta) \equiv 0$, $Z^*(t, 0, \eta) \equiv 0$.

9. Partial stability under large initial perturbations. Consider the following system of differential equations of perturbed motion

$$\mathbf{x}^\cdot = X(t, \mathbf{x}), \quad \mathbf{x} = (\mathbf{y}, \mathbf{z}) \tag{9.1}$$

where the right-hand sides satisfy the general conditions of /2/.

It is shown in /13, 14/ (see also /2/) that if there exists a function V for system (9.1) such that

$$a(\| \mathbf{y} \|) \leqslant V(t, \mathbf{y}, \mathbf{z}) \leqslant b(\| \mathbf{y} \|), \quad V(t, 0, 0) \equiv 0, \quad V^\cdot \leqslant 0 \tag{9.2}$$

($a(r)$, $b(r)$ are continuous, monotonically increasing functions of $r \in [0, H]$, $a(0) = b(0) = 0$), then the motion $\mathbf{x} = 0$ has the following property: for any $\varepsilon, t_0 \geqslant 0$ there exists $\delta(\varepsilon) > 0$ such that, if $\|\mathbf{y}_0\| < \delta, \|\mathbf{z}_0\| < \infty$, then $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geqslant t_0$.

In applied problems, the condition $\|\mathbf{z}_0\| < \infty$ may be replaced by $\|\mathbf{z}_0\| < \Delta$, where $\Delta > 0$ is a given number.

Definition 2. Let $\Delta > 0$ be a given number. The motion $\mathbf{x}=0$ of system (9.1) is said to be $\mathbf{y}$-stable for large $\mathbf{z}_0$ if, for any $\varepsilon, t_0 \geqslant 0$, there exists $\delta(\varepsilon) > 0$ such that, if $\|\mathbf{y}_0\| < \delta, \|\mathbf{z}_0\| < \Delta$, then $\|\mathbf{y}(t; t_0, \mathbf{x}_0)\| < \varepsilon$ for all $t \geqslant t_0$.

We shall show that investigation of $\mathbf{y}$-stability for large $\mathbf{z}_0$ can be conducted with weaker conditions imposed on the Lyapunov function than in /13, 14/.

Theorem 4. Suppose there exists a function V for system (9.1) such that

$$a(\|\mathbf{y}\|) \leqslant V(t, \mathbf{y}, \mathbf{z}) \leqslant b(\|\mathbf{x}\|) \\ V(t, 0, 0) \equiv V(t, 0, \mathbf{z}) \equiv 0, \quad V^{\cdot} \leqslant 0 \tag{9.3}$$

then the motion $\mathbf{x} = 0$ is $\mathbf{y}$-stable for large $\mathbf{z}_0$.

Proof. In the domain $t \geqslant 0$, $\|\mathbf{x}\| < L = \text{const} < \infty$, the function V is bounded: $V \leqslant b(\|\mathbf{x}\|)$. Therefore, for any $\varepsilon > 0$, $t_0 \geqslant 0$, since $V(t, 0, 0) \equiv V(t, 0, \mathbf{z}) \equiv 0$, there exists $\delta(\varepsilon) > 0$ such that, if $\|\mathbf{y}_0\| < \delta$, $\|\mathbf{z}_0\| < \Delta$, then for all $t_0 \geqslant 0$ one has $V(t_0, \mathbf{x}_0) < a(\varepsilon)$. For any solution $\mathbf{x}(t) = \mathbf{x}(t; t_0, \mathbf{x}_0)$ with $\|\mathbf{y}_0\| < \delta$, $\|\mathbf{z}_0\| < \Delta$, we see that, since $V^{\cdot} \leqslant 0$ (see /2/)

$$a(\|\mathbf{y}(t; t_0, \mathbf{x}_0)\|) \leqslant V(t, \mathbf{x}(t; t_0, \mathbf{x}_0)) \leqslant V(t_0, \mathbf{x}_0) < a(\varepsilon), \quad t \geqslant t_0$$

whence, using the properties of the function $a(r)$, we conclude that $\|\mathbf{y}(t; t_0; \mathbf{x}_0)\| < \varepsilon$, $t \geqslant t_0$. This completes the proof.

Remarks. 1°. Conditions (9.3) are weaker than (9.2); at the same time, $\mathbf{y}$-stability for large $\mathbf{z}_0$ is, practically speaking, equivalent to $\mathbf{y}$-stability in the large relative to $\mathbf{z}_0$ as defined in /13, 14/.

2°. The results of /13, 14/ and Theorem 4 are extensions of a theorem of Rumyantsev /1/ on $\mathbf{y}$-stability.

Example. Following the analysis of /15, 16/, we consider the motion of a point of unit mass in a constant gravitational field on the surface $x_1 = 0.5x_2^2(1 + x_3^2)$ in a coordinate frame $Ox_1x_2x_3$ with Ox_1 pointing vertically upwards. The kinetic energy T and potential energy Π are

$$T = \tfrac{1}{2}\{x_3^{\cdot 2} + x_2^{\cdot 2} + x_2^2[x_2^{\cdot}(1 + x_3^2) + x_3^{\cdot}x_3x_2]^2\} \\ \Pi = \tfrac{1}{2}gx_2^2(1 + x_3^2), \quad g = \text{const} > 0$$

If $\mathbf{y} = (x_2, x_2^{\cdot}, x_3^{\cdot})$, $z = x_3$, then the function $V = T + \Pi$ satisfies conditions (9.3), and so by Theorem 4, the equilibrium position $x_i = x_i^{\cdot} = 0$, $(i = 1, 2, 3)$ of the point is $\mathbf{y}$-stable for large z_0. At the same time, the condition $V \leqslant b(\|\mathbf{y}\|)$ is not satisfied for this function V and the results of /13, 14/ are not applicable.

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A THEORY OF LINEAR NON-CONSERVATIVE SYSTEMS*

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Linear systems with non-conservative positional forces are considered. It is proved that Rayleigh's theorem on the behaviour of the natural frequencies of conservative systems when the rigidity and inertia are varied cannot be generalized to such systems. A necessary and sufficient condition is established under which unstable non-conservative systems can be stabilized by dissipative forces of a special type.

It is shown that in the case of forced harmonic oscillations at frequencies lying beyond the spectrum of the corresponding conservative system, the application of non-conservative forces diminishes the absolute value of the action functional. Least upper bounds are obtained for the amplitudes of the forced oscillations, independent of the non-conservative forces.

1. The free oscillations of a system with non-conservative positional forces are described by the equation

$$\begin{gathered} M\mathbf{x}^{\cdot\cdot} + A\mathbf{x} = 0 \\ A = C + K, \quad M = \| m_{ij} \|_1^n, \quad C = \| c_{ij} \|_1^n, \quad K = \| k_{ij} \|_1^n, \\ \mathbf{x} = (x_1, \ldots, x_n)^T \end{gathered} \tag{1.1}$$

where $\mathbf{x}$ is the vector of generalized coordinates, M and C are the symmetric inertia and elasticity matrices and K is the skew-symmetric matrix of non-conservative forces.

By Rayleigh's Theorem /1/, the frequencies of the natural oscillations of the corresponding conservative system $(K = 0)$ increase (do not decrease) as the rigidity increases and as the inertia of the system decreases. Zhuravlev has generalized this theorem to systems with gyroscopic forces /2/. He has suggested the following problem: is the analogous propostion true for system (1.1) when the non-conservative forces are sufficiently small? Below we shall answer this question in the negative.

We may assume without loss of generality that $M = E$ is the unit matrix. Let λ_i be a simple real eigenvalue of A, $\mathbf{a}_i$ a corresponding eigenvector and $\mathbf{b}_i$ an eigenvector of the transposed matrix A^T corresponding to λ_i. In general, the vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ are linearly independent; we shall assume henceforth that this is indeed the case. Since $(\mathbf{a}_i, \mathbf{b}_i) \neq 0$ (where the parentheses denote the scalar product) /3/, we may assume that $(\mathbf{a}_i, \mathbf{b}_i) = 1$.

Put $C(\varepsilon) = C_0 + \varepsilon C_1$ in (1.1), where C_1 is a symmetric positive definite matrix. Let us investigate the behaviour of $\lambda_i(\varepsilon)$ as ε increased. We shall show that, unlike the conservative case, $\lambda_i(\varepsilon)$ is a decreasing function of ε when C_1 is suitably chosen.

As we know,

$$\delta_i = d\lambda_i(\varepsilon)/d\varepsilon\,|_{\varepsilon=0} = (\mathbf{a}_i, C_1\mathbf{b}_i) \tag{1.2}$$

Putting $\mathbf{a}_i = \mathbf{c}_i + \mathbf{d}_i$, $\mathbf{b}_i = \mathbf{c}_i - \mathbf{d}_i$ and using the symmetry of C_1, we obtain

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